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GENERALIZATION AND DEFORMATIONS OF QUANTUM GROUPS; QUANTIZATION OF ALL SIMPLE LIE BI-ALGEBRAS

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Abstract.

A large family of "standard" coboundary Hopf algebras is investigated. The existence of a universal R-matrix is demonstrated for the case when the parameters are in general position. Special values of the parameters are characterized by the appearance of certain ideals; in this case the universal R-matrix exists on the associated algebraic quotient. In special cases the quotient is a "standard" quantum group; all familiar quantum groups including twisted ones are obtained in this way. In other special cases one finds new types of coboundary bi-algebras.

A large class of first order deformations of all these standard bi-algebras is investigated and the associated deformed universal R-matrices have been calculated. One obtains, in particular, universal R-matrices associated with all simple, complex Lie algebras (classification by Belavin and Drinfeld) to first order in the deformation parameter.

1. Introduction.

Quantum groups sprouted in that fertile soil where mathematics overlaps with physics. The mathematics of quantum groups is exciting, and the applications to physical modelling are legion. It is the more surprising that some aspects of the structure of quantum groups remain to be explored; this is especially true of those aspects that bear upon the problem of classification. The quantum groups that have so far found employment in physics are very special (characterized by a single "deformation" parameter q). It is true that these applications are susceptible to some generalization, by the process of twisting; unfortunately it is easy to receive the totally erroneous impression that twisting is a gauge transformation that relates equivalent structures. The fact that twisted or multiparameter quantum groups differ qualitatively among themselves becomes evident when one investigates their rigidity to deformation. Deformation theory is a means of attacking the classification problem; at the same time it offers a wider horizon against which to view the whole subject. The new quantum groups discovered this way (the deformations of the twisted ones) are dramatically different; the physical applications should be of a novel kind.

Let $\mathcal{L}$ be a simple Lie algebra over $\mathbb{C}$. A structure of coboundary Lie bialgebra on $\mathcal{L}$ is determined by a "classical" r-matrix; an element $r \in \mathcal{L} \otimes \mathcal{L}$ that satisfies the classical Yang-Baxter relation

$$[r_{12}, r_{13} + r_{23}] + [r_{13}, r_{23}] = 0 \ , \tag{1.1}$$

as well as the symmetry condition

$$r + r^t = \hat{K} \ , \tag{1.2}$$

where $\hat{K}$ is the Killing form of $\mathcal{L}$. The classification of r-matrices of simple complex Lie algebras was accomplished by Belavin and Drinfeld [BD].

It is widely believed that there corresponds, to each such r-matrix, via a process of "quantization," a unique quantum group [D2]. Somewhat more preciesely, one expects that there exists a Hopf algebra deformation $\tilde{U}(\mathcal{L})$ of $U(\mathcal{L})$, and an element $R \in \tilde{U}(\mathcal{L}) \otimes \tilde{U}(\mathcal{L})$ such that $\Delta R = R\Delta'$, where Δ is the coproduct of $\tilde{U}(\mathcal{L})$ and Δ' is the opposite coproduct, satisfying the (quantum) Yang-Baxter relation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \ ; \tag{1.3}$$

such that r can be recovered by an expansion of R with respect to a parameter $\hbar$:

$$R = 1 + \hbar r + o(\hbar^2) \ . \tag{1.4}$$

Till now, this program has been realized for r-matrices of a restricted class that we shall call "standard".

Definition 1. Let $\mathcal{L}$ be a simple, complex Lie algebra, $\mathcal{L}^0$ a Cartan subalgebra and Δ^+ a set of positive roots. A standard r-matrix for $\mathcal{L}$ has the expression

$$r = r_0 + \sum_{\alpha \in \Delta^+} E_{-\alpha} \otimes E_\alpha \ . \tag{1.5}$$

Here $r_0 \in \mathcal{L}^0 \otimes \mathcal{L}^0$ is restricted by (1.2).

The (universal) R-matrix that corresponds to a standard r-matrix is known. An explicit formula has been given for the simplest choice of r_0 [KR]. Of the more or less explicit formulas for those R-matrices that correspond to the standard r-matrices in general, there is one that seems the more fundamental [Ro][LSo]

$$R = R^0 \big(1 + \sum_\alpha e_{-\alpha} \otimes e_\alpha + \ldots\big) \ . \tag{1.6}$$

Here $\{H_a, e_\alpha, e_{-\alpha}\}$ are Chevalley-Drinfeld generators associated with the Cartan subalgebra and simple roots, R^0 involves only the former. An R-matrix of this form will be called standard; a precise definition (in a more general context) will be given in Section 2, Definition 2.2. The relationship between (1.5) and (1.6) will be examined in Section 11.

The R-matrices associated with the twisted quantum groups discovered by Reshetikhin [R] and others [Sc][Su] are thus all included in the rubrique "standard". The principal characteristic of a standard R-matrix is that it "commutes with Cartan":

$$\big[H_a \otimes 1 + 1 \otimes H_a, R\big] = 0.$$

Non-standard R-matrices are known only in the fundamental representation [CG][FG].

The original plan of this work was to use deformation theory to obtain some information about the so far unknown quantum groups that are alleged to be associated with

non-standard r-matrices. This seems a reasonable approach because (i) the non-standard r-matrices can be viewed, and effectively calculated [F], as deformations of standard r-matrices and (ii) the largest family of non-standard quantum groups known so far was found by applying deformation theory to certain standard R-matrices in the fundamental representation [FG2].

However, the application of deformation theory to the evaluation of non-standard R-matrices turns out to be complicated, perhaps because no useful cohomological framework could be discovered. If some progress has been achieved in the present paper, then it is due, in the first place, to the idea of focusing on the representation (1.6) of the standard universal R-matrix, and in the second place to the discovery of a differential complex associated with the Yang-Baxter relation: a detailed study of (1.6) turned out to be unexpectedly rewarding.

It turns out that the representation (1.6) for an R-matrix that satisfies the Yang-Baxter equation makes sense in a context that is much wider than quantum groups.

We introduce an algebra $\mathcal{A}$ (Definition 2.1) with generators $\{H_a, e_\alpha, e_{-\alpha}\}$ that satisfy certain relations, including the following (see also Eq. (1.7b) below):

$$[H_a, H_b] = 0,\ [H_a, e_{\pm\alpha}] = \pm H_a(\alpha) e_{\pm\alpha}\ , \tag{1.7a}$$

with $H_a(\alpha) \in \mathbb{C}$. We define a standard R-matrix on $\mathcal{A}$ – Definition 2.2. – as a formal series of the form

$$R = \exp(\varphi^{ab} H_a \otimes H_b)\big(1 + e_{-\alpha} \otimes e_\alpha + \sum_{k=2}^{\infty} t_{(\alpha)}^{(\alpha')} e_{-\alpha_1} \dots e_{-\alpha_k} \otimes e_{\alpha'_1} \dots e_{\alpha'_k}\big)\ ,$$

with parameters $\varphi^{ab} \in \mathbb{C}$, fixed, and try to determine the coefficients $t_{(\alpha)}^{(\alpha')} \in \mathbb{C}$ so that the Yang-Baxter relation (1.3) is satisfied. One finds that this requires additional relations, namely

$$[e_\alpha, e_{-\beta}] = \delta_\alpha^\beta \big(e^{\phi(\alpha,\cdot)} - e^{-\varphi(\cdot,\alpha)}\big)\ , \tag{1.7b}$$

with $\varphi(\alpha,\cdot) = \varphi^{ab} H_a(\alpha) H_b$, $\varphi(\cdot,\alpha) = \varphi^{ab} H_a H_b(\alpha)$. These relations are therefore included in the definition of the algebra $\mathcal{A}$, Definition (2.1). Generically, with the parameters in general position, no further relations are required.

The generators H_a of $\mathcal{A}$ generate an Abelian subalgebra that is denoted $\mathcal{A}^0$ and sometimes called the Cartan subalgebra. A key point is to refrain from introducing,

a priori, any (generalized) Serre relations among the Chevalley-Drinfeld generators e_α, or among the $e_{-\alpha}$. The algebras of ultimate interest are obtained subsequently, by identifying an appropriate ideal $I \subset \mathcal{A}$ that intersects $\mathcal{A}^0$ trivially, and passing to the quotient $\mathcal{A}/I$. This is the strategy of Chevalley [C], fully exploited in the theory of Kac-Moody algebras [K][M]; here it is applied to "generalized quantum groups."

The first result is Theorem 2. It asserts that the Yang-Baxter relation for the standard R-matrix on $\mathcal{A}$ is equivalent to a simple, linear recursion relation for the coefficients $t^{(\alpha')}_{(\alpha)}$.

The integrability of this recursion relation, Eq.(2.14), is studied in Sections 3, 4 and 5; it is related to the existence of "constants," and eventually to generalized Serre relations. Generically, there are no constants and no obstructions, whence the second result that, when the parameters of $\mathcal{A}$ are in general position, there exists a unique set of coefficients $t^{(\alpha')}_{(\alpha)}$ such that the standard R-matrix satisfies the Yang-Baxter relation.

Constants exist on certain hyper-surfaces in the space of parameters of $\mathcal{A}$; they represent obstructions to the solution of the recursion relation (2.14) and thus to the Yang-Baxter relation. Constants are studied in a slightly more general context in Sections 3 and 4; their complete classification seems to present a formidable, but not unsolvable problem. The relevance of this discussion to Yang-Baxter is demonstrated in Section 5, and the proof of Theorem 2 can now be completed in Section 6.

The study of the obstructions is taken up again in Section 7. The third main result is Theorem 7: the obstructions (that is, the constants) generate an ideal $I \subset \mathcal{A}$, and a unique standard R-matrix, satisfying Yang-Baxter, exists on $\mathcal{A}/I$.

In Sections 8-10 we turn to the deformations of the standard R-matrix, but in a context that is wider than quantum groups. We calculate a class of first order deformations of the standard R-matrix on $\mathcal{A}/I$ for any ideal of obstructions $I \subset \mathcal{A}$. This is our fourth result, Theorem 10. The main difficulty is that the problem is not well posed, for we have been unable to discover a category that is both natural and convenient in which to calculate <u>all</u> deformations. We limit our study to a class of deformations. The good news is Theorem 11: when we specialize to the case of simple quantum groups, then we obtain quantizations of all simple Lie bialgebras (constant r-matrices) of Belavin and Drinfeld.

Finally, in Section 12, we define the coproduct, counit and antipode that turn all these algebras into Hopf algebras.

2. Standard Universal R-matrices.

The universal R-matrix of a standard or twisted quantum group has the form

$$\begin{aligned} R = &\exp(\varphi^{ab} H_a \otimes H_b) \\ &\times \big(1 + t_\alpha(e_{-\alpha} \otimes e_\alpha) + t_{\alpha\beta}(e_{-\alpha}e_{-\beta} \otimes e_\alpha e_\beta) + t'_{\alpha\beta}(e_{-\alpha}e_{-\beta} \otimes e_\beta e_\alpha) + \ldots\big) \ . \end{aligned} \tag{2.1}$$

The H_a are generators of the Cartan subalgebra, the e_α are generators associated with simple roots, $\varphi^{ab},\ t_\alpha,\ t_{\alpha\beta},\ t'_{\alpha\beta},\ \ldots$ are in the field; the unwritten terms are monomials in the e_α and $e_{-\alpha}$.

More generally, consider the expression (2.1) in which $H_a, e_{\pm\alpha}$ generate an associative algebra with certain relations.

Definition 2.1. Let M, N be two countable sets, φ, ψ two maps,

$$\begin{aligned} \varphi: \quad & M \otimes M \to \mathbb{C}\ , \qquad a, b \to \varphi^{ab}\ , \\ \psi: \quad & M \otimes N \to \mathbb{C}\ , \qquad a, \beta \to H_a(\beta)\ . \end{aligned} \tag{2.2}$$

Let $\mathcal{A}$ or $\mathcal{A}(\varphi, \psi)$ be the universal, associative, unital algebra over $\mathbb{C}$ with generators $\{H_a\}\, a \in M,\ \{e_{\pm\alpha}\}\, \alpha \in N$, and relations

$$[H_a, H_b] = 0\ , \quad [H_a, e_{\pm\beta}] = \pm H_a(\beta) e_{\pm\beta}\ , \tag{2.3}$$

$$[e_\alpha, e_{-\beta}] = \delta^\beta_\alpha \big(e^{\varphi(\alpha,\cdot)} - e^{-\varphi(\cdot,\alpha)}\big)\ , \tag{2.4}$$

with $\varphi(\alpha,\cdot) = \varphi^{ab} H_a(\alpha) H_b,\ \varphi(\cdot,\alpha) = \varphi^{ab} H_a H_b(\alpha)$ and $e^{\varphi(\alpha,\cdot)+\varphi(\cdot,\alpha)} \neq 1,\ \ \alpha \in N$. (The last condition on the parameters is included in order to avoid having to make some rather trivial exceptions.)

The free subalgebra generated by $\{e_\alpha\}\ \alpha \in N$ (resp. $\{e_{-\alpha}\}\ \alpha \in N$) will be denoted $\mathcal{A}^+$ (resp. $\mathcal{A}^-$); these subalgebras are Z^+-graded, the generators having grade 1. The subalgebra generated by $\{H_a\}\, a \in M$ is denoted $\mathcal{A}^0$.

Definition 2.2. A standard R-matrix is a formal series of the form

$$R = \exp\bigl(\varphi^{ab} H_a \otimes H_b\bigr)\bigl(1 + e_{-\alpha}\otimes e_\alpha + \sum_{k=2}^{\infty} t_{\alpha_1\ldots\alpha_k}^{\alpha'_1\ldots a'_k} e_{-\alpha_1}\ldots e_{-\alpha_k}\otimes e_{\alpha'_1}\ldots e_{\alpha'_k}\bigr)\ . \tag{2.5}$$

In this formula, and in others to follow, summation over repeated indices is implied. For fixed $(\alpha) = \alpha_1, \ldots, \alpha_k$ the sum over (α') runs over the permutations of (α). The coefficients $t_{(\alpha)}^{(\alpha')}$ are in $\mathbb{C}$.

The special property associated with the qualification "standard" is that "R commutes with Cartan"; indeed $[R,\ H_a \otimes 1 + 1 \otimes H_a] = 0,\ a \in M$.

We shall determine under what conditions on the parameters $\varphi^{ab},\ H_a(\beta)$ of $\mathcal{A}$, and for what values of the coefficients $t_{(\alpha)}^{(a')}$, the R-matrix (2.5) satisfies the Yang-Baxter relation

$$YB\ :=\ R_{12}R_{13}R_{23} - R_{23}R_{13}R_{12} = 0\ . \tag{2.6}$$

This expression is a formal series in which each term has the form $\psi_1\otimes\psi_2\otimes\psi_3 \in \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}$. We assign a double grading as follows. First extend the grading of $\mathcal{A}^+$ to the subalgebra of $\mathcal{A}$ that is generated by $\{H_a\}\, a \in M$ and $\{e_\alpha\}\, \alpha \in N$, by assigning grade zero to H_a, and similarly for $\mathcal{A}^-$. Then ψ_1 and ψ_3 (but not ψ_2) belong to graded subalgebras of $\mathcal{A}$. If ψ_1 and ψ_3 have grades ℓ and n, respectively, then define

$$\text{grade}\ (\psi_1\otimes\psi_2\otimes\psi_3) = (\ell, n)\ . \tag{2.7}$$

To give a precise meaning to (2.6) we first declare that we mean for this relation to hold for each grade (ℓ, n) separately. This is not enough, for the number of terms contributing to each grade is infinite in general. The appearance of exponentials in the H_a can be dealt with in the same way as in the case of simple quantum groups [V]. If the sets M, N are infinite, then all results are basis dependent. Eq.(2.6) means that YB, projected on any finite subalgebra of $\mathcal{A}$, vanishes on each grade; the statement thus involves only finite sums. The analysis of (2.6) will be organized by ascending grades.

Remarks. (i) It is an immediate consequence of (2.6), in grade (1,1), that

$$[e_\alpha, e_{-\beta}] = \delta_\alpha^\beta\bigl(e^{\varphi(\alpha,\cdot)} - e^{-\varphi(\cdot,\alpha)}\bigr)\ . \tag{2.8}$$

This relation was therefore included in the definition of the algebra $\mathcal{A}$. (ii) No relations of the Serre type have been imposed; in fact no relations whatever on the subalgebras $\mathcal{A}^+$ and $\mathcal{A}^-$, freely generated respectively by the e_α and the $e_{-\alpha}$. The contextual meaning of such relations, including relations of the Serre type, will be discussed in Sections 3-5 and especially in Section 7.

Before stating the main result, it will be convenient to show the direct evaluation of YB up to grade (2,2). We expand

$$R^0 \ := \ \exp(\varphi^{ab} H_a \otimes H_b) = R^i \otimes R_i \ , \tag{2.9}$$

sums over a, b, i implied. Then

$$e_{-\alpha} R^i \otimes R_i = R^i e_{-\alpha} \otimes e^{\varphi(\alpha,\cdot)} R_i \ . \tag{2.10}$$

Grade (1,1). The contributions to $R_{12}R_{13}R_{23}$ are of two kinds:

$$R^i R^j e_{-\alpha} \otimes R_i R^k \otimes R_j e_\alpha R_k \ ,$$
$$R^i e_{-\alpha} R^j \otimes R_i e_\alpha R^k e_{-\beta} \otimes R_j R_k e_\beta \ .$$

Cancellation in YB is equivalent to Eq.(2.4).

Grade (1,2). The contributions to $R_{12}R_{13}R_{23}$ are

$$R^i R^j e_{-\beta} \otimes R_i R^k e_{-\alpha} \otimes R_j e_\beta R_k e_\alpha \ ,$$
$$t^{\alpha'\beta'}_{\alpha\beta} R^i e_{-\gamma} R^j \otimes R_i e_\gamma R^k e_{-\alpha} e_{-\beta} \otimes R_j R_k e_{\alpha'} e_{\beta'} \ .$$

Cancellation in YB requires that

$$\begin{aligned} t^{\alpha\beta}_{\alpha\beta} &= (1 - e^{-\varphi(\alpha,\beta)-\varphi(\beta,\alpha)})^{-1} \ , \\ t^{\beta\alpha}_{\alpha\beta} &= -e^{-\varphi(\beta,\alpha)} t^{\alpha\beta}_{\alpha\beta} \ , \quad \alpha \neq \beta \ , \\ t^{\alpha\alpha}_{\alpha\alpha} &= (1 + e^{-\varphi(\alpha,\alpha)})^{-1} \ . \end{aligned} \tag{2.11}$$

These conditions are necessary and sufficient that the standard R-matrix satisfy (2.6) up to grade (2,2).

The obstructions to the existence of coefficients $t^{(\alpha')}_{(\alpha)}$ such that $YB = 0$ up to grade (2,2) are therefore as follows:

$$\begin{aligned} &1 + e^{-\varphi(\alpha,\alpha)} = 0 \ \text{ for some } \alpha \in N \ , \\ &1 - e^{-\varphi(\alpha,\beta)-\varphi(\beta,\alpha)} = 0 \ \text{ for some pair } \alpha \neq \beta \ . \end{aligned} \tag{2.12}$$

They are typical of obstructions encountered at all grades.

Let

$$t_{\alpha_1\ldots\alpha_l} = t^{\alpha'_1\ldots\alpha'_\ell}_{\alpha_1\ldots\alpha_\ell}\, e_{\alpha'_1}\ldots e_{\alpha'_\ell}\ . \tag{2.13}$$

Theorem 2. The standard R-matrix (2.5), on $\mathcal{A}$, satisfies the Yang-Baxter relation (2.6) if and only if the coefficients $t^{(\alpha')}_{(\alpha)}$ satisfy the following recursion relation

$$[t_{\alpha_1\ldots\alpha_\ell}, e_{-\gamma}] = e^{\varphi(\gamma,\cdot)}\delta^\gamma_{\alpha_1} t_{\alpha_2\ldots\alpha_\ell} - t_{\alpha_1\ldots\alpha_{\ell-1}}\delta^\gamma_{\alpha_\ell} e^{-\varphi(\cdot,\gamma)}\ . \tag{2.14}$$

Proof. (First part.) We shall prove that (2.14) is necessary – the "only if" part. Then we shall study the integrability of (2.14). Later, in Section 6, we shall complete the proof of Theorem 2. Insert (2.5) into (2.6) and use (2.10). The contribution to *YB* in grade (ℓ, n) is

$$R^0_{12}R^0_{13}R^0_{23}e_{-\alpha_1}\ldots e_{-\alpha_\ell} \otimes P^{\gamma_1\cdots\gamma_n}_{\alpha_1\ldots\alpha_\ell} \otimes e_{\gamma_1}\ldots e_{\gamma_n}\ ,$$

in which P is the sum over m, from 0 to $\min(\ell, n)$, of the following elements of $\mathcal{A}$,

$$\begin{aligned} &t^{\gamma_1\cdots\gamma_m}_{\alpha_{\ell-m+1}\ldots\alpha_\ell} t_{\alpha_1\ldots\alpha_{\ell-m}} e^{-\varphi(\cdot,\sigma)} t^{\gamma_{m+1}\cdots\gamma_n} \\ &\qquad - t^{\gamma_{n-m+1}\cdots\gamma_n}_{\alpha_1\ldots\alpha_m} t^{\gamma_1\cdots\gamma_{n-m}} e^{\varphi(\tau,\cdot)} t_{\alpha_{m+1}\ldots\alpha_\ell}\ , \end{aligned} \tag{2.15}$$

where $\sigma = \gamma_1 + \ldots + \gamma_m$ and $\tau = \alpha_1 + \ldots + \alpha_m$. The Yang-Baxter relation is satisfied in grade (ℓ, n) if and only if this expression, summed over m, vanishes for every index set $\alpha_1, \ldots, \alpha_\ell$ and $\gamma_1, \ldots, \gamma_n$. This is so because $\mathcal{A}^+$ and $\mathcal{A}^-$ are freely generated. We have used the definition (2.13) and

$$t^{\gamma_1\cdots\gamma_n} \ := \ t^{\gamma_1\cdots\gamma_n}_{\gamma'_n\cdots\gamma'_n} e_{-\gamma'_1}\ldots e_{-\gamma'_n}\ . \tag{2.16}$$

The lowest grades in which $t_\ell = (t_{\alpha_1\ldots\alpha_\ell})$ appears are $(\ell, 0)$ and $(0, \ell)$. In these cases $m = 0$ and (2.15) vanishes identically. At grade $(\ell, 1)$ one finds (summing $m = 0, 1$), the linear recursion relation

$$[t_\ell, e_{-\gamma}] = e^{\varphi(\gamma,\cdot)}\delta^\gamma_{\alpha_1} t_{\ell-1} - t_{\ell-1}\delta^\gamma_{\alpha_\ell} e^{-\varphi(\cdot,\gamma)}\ , \tag{2.17}$$

the full expression for which is Eq. (2.14). This equation is therefore necessary. That it is also sufficient will be proved in Section 6.

3. Differential Algebras.

Let B be the unital $\mathbb{C}$-algebra freely generated by $\{\xi_i\}$ $i \in N$, countable. Suppose given a map

$$q : N \times N \to \mathbb{C} \ , \quad (i,j) \to q^{ij} \neq 0 \ . \tag{3.1}$$

Introduce the natural grading on B, $B = \bigoplus B_n$, and a set of differential operators

$$\partial_i : \ B_n \to B_{n-1} \ , \quad i \in N \ , \tag{3.2}$$

defined by

$$\partial_i \xi_j = \delta_i^j + q^{ij} \xi_j \partial_i \ . \tag{3.3}$$

We study the problem of integrating sets of equations of the type:

$$\partial_i X = Y_i \ , \quad X \in B \ , \quad Y_i \in B \ , \quad i \in N \ . \tag{3.4}$$

The collection $\{Y_i\}$ $i \in N$ can be interpreted as the components of a B-valued one-form Y, on the space $\{c^i \partial_i \ , \ \ c^i \in \mathbb{C} \ , \ \ i \in N\}$. A constant in B_n is an element $X \in B_n$, $\partial_i X = 0$.

Proposition 3. (a) The following statements are equivalent: (i) Eq. (3.4) is integrable for every one-form Y with components in B_{n-1}. (ii) There are no constants in B_n. (b) When the parameters q^{ij} are in general position, then there are no constants in B_n, $n \geq 1$.

Proof. Let

$$X = X^{i_1 \ldots i_n} \xi_{i_1} \ldots \xi_n \ \in B_n,$$

then $\partial_i X = 0$ means that, for each index set,

$$\begin{aligned} X^{i_1 \ldots i_n} + q^{i_1 i_2} X^{i_2 i_1 i_3 \ldots} + q^{i_1 i_2} q^{i_1 i_3} X^{i_2 i_3 i_1 i_4 \ldots} \\ + \ldots + q^{i_1 i_2} \ldots q^{i_1 i_n} X^{i_2 \ldots i_n i_1} = 0 \ . \end{aligned} \tag{3.5}$$

Now fix the unordered index set $\{i_1, \ldots, i_n\}$. If the values are distinct then we have a set of $n!$ equations for $n!$ coefficients; in general the number of equations is always equal to the number of unknowns. Solutions exist if and only if the determinant of the matrix of coefficients vanishes. This determinant is an algebraic function of the q^{ij} so that solutions of (3.5), other than $X = 0$, exist only on an algebraic subvariety of parameter space.

The calculation of all these determinants appears to be a formidable problem. For $n = 2$ the result is

$$D^{12} = 1 - q^{12}q^{21} \ , \quad D^{11} = 1 + q^{11} \ . \tag{3.6}$$

For $n = 3$,

$$\begin{aligned} D^{123} &= (1-\sigma^{12})(1-\sigma^{13})(1-\sigma^{23})(1-\sigma^{12}\sigma^{13}\sigma^{23}) \ , \\ D^{112} &= (1+q^{11})(1-\sigma^{12})(1-q^{11}\sigma^{12}) \ , \\ D^{111} &= 1 + q^{11} + (q^{11})^2 \ , \quad \sigma^{12} \ := \ q^{12}q^{21} \ . \end{aligned} \tag{3.7}$$

It is natural to pass from B to the quotient by the ideal generated by the constants. In B_2 the constants are

$$\xi_1\xi_2 - q^{21}\xi_2\xi_1 \ , \quad \text{when} \quad \sigma^{12} = 1 \ , \tag{3.8}$$

$$\xi_1\xi_1 \qquad\qquad , \quad \text{when} \quad q^{11} = -1 \ . \tag{3.9}$$

If $q^{ii} = -1, \ i \in N$ and $\sigma^{ij} = 1, \ i \neq j$, then the quotient is a q-Grassmann algebra or quantum antiplane. The constants in B_3 are

$$\xi_1\xi_1\xi_1 \qquad , \quad 1 + q^{11} + (q^{11})^2 = 0 \ , \tag{3.10}$$

$$\xi_1\xi_1\xi_2 - (q^{21})^2 \ \xi_2\xi_1\xi_1 \qquad , \quad 1 + q^{11} = 0 \ , \tag{3.11}$$

$$q^{12}\xi_1\xi_1\xi_2 - (1+\sigma^{12}) \ \xi_1\xi_2\xi_1 + q^{21}\xi_2\xi_1\xi_1 \ , \quad q^{11}\sigma^{12} = 1 \ ; \tag{3.12}$$

if $\sigma^{12} = 1$, there are two constants

$$q^{11}\xi_1\xi_1\xi_2 - (1+q^{11})\xi_1\xi_2\xi_1 + (q^{21})^2 \ \xi_2\xi_1\xi_1 \ , \tag{3.13}$$

$$[[\xi_1,\xi_2]_{q^{21}},\xi_3]_{q^{31}q^{32}} \ , \quad [a,b]_q \ := \ ab - qba \ , \tag{3.14}$$

and finally if $\sigma^{12}\sigma^{13}\sigma^{23} = 1$ there is one,

$$\left(\frac{1}{q^{31}} - q^{13}\right)(\xi_1\xi_2\xi_3 + q^{31}q^{32}q^{21}\xi_3\xi_2\xi_1) + \text{cyclic.} \tag{3.15}$$

Annulment of (3.8), (3.12) and (3.13) are q-deformed Serre relations [D1]. The last item, Eq. (3.15), may be something new; it should be interesting to study the quotient of the algebra B with 3 generators by the ideal generated by this constant.

A constant that involves only one variable, ξ_1 say, exists if and only if q^{11} is a root of unity,

$$\xi_1^n \ \ \text{constant} \ \ \text{iff} \ \ (q^{11})^n = 1, \quad n = 2, 3, \ldots .$$

It is easy to determine all constants of the q-Serre type; that is, all those that involve two geneators and one linearly,

$$C := \sum_{m=0}^{k} Q_m^k \ (\xi_1)^m \xi_2 (\xi_1)^{k-m} \ = 0. \tag{3.16}$$

With $q = q^{11}$,

$$\partial_1 (\xi_1)^m = (m)_q \ (\xi_1)^{m-1} \ , \quad (m)_q \ := \ 1 + q + \ldots + q^{m-1} \ . \tag{3.17}$$

Setting $\partial_1 C = 0$ gives, for $q^n \neq 1, \ n \in \mathbb{Z}$,

$$Q_m^k = (-q^{12})^m \ q^{m(m-1)/2} \begin{pmatrix} k \\ m \end{pmatrix}_q \ , \tag{3.18}$$

while $\partial_2 C = 0$ is equivalent to

$$\prod_{m=0}^{k-1} (1 - q^m \sigma^{12}) = 0 \ . \tag{3.19}$$

When $k = 2$, compare D^{112} in Eq. (3.7). If k is the smallest integer such that a relation like (3.16) holds, then

$$1 - q^{k-1} \sigma^{12} = 0 \ . \tag{3.20}$$

Here are some partial results for B_4 and B_5. D_{1234} is the product of 12 factors of the form $1 - \sigma_{ij}$, 4 factors of the form $1 - \sigma_{ij}\sigma_{kl}\sigma_{mn}$, 2 identical factors of the form $1 - \sigma_{12} \ldots \sigma_{34}$; each group accounts for 24 orders in the q's. D_{12345} is the product of 60 factors of the first type, 20 factors of the second type, 10 factors of the third type and 6 identical factors of the form $1-$(product of all ten $\sigma_{ij}, i \neq j$); each group acconts for 120 orders in the q's.

4. Differential Complexes.

In the generic case, when there are no constants in B_n, the equation $\partial_i X = Y_i,\ Y_i \in B_{n-1},\ i \in N$, is always solvable, for any one-form Y. All one-forms are exact, to be closed has no meaning and the differential complex is highly trivial.

The existence of a constant $C \in B_n$ implies that there are one-forms valued in B_{n-1} that are not exact. To each 1-dimensional space of constants in B_n there is a one-dimensional space of non-exact one-forms, valued in B_{n-1}, defined modulo exact one-forms and obtainable as a limit of $\partial_i X$ as $X \to C$, after factoring out a constant. Thus,

$$X = \xi_1\xi_2 - q^{21}\xi_2\xi_1 \tag{4.1}$$

becomes a constant as $\sigma^{12} \to 1$, and a representative of the associated class of non-exact one-forms is given by

$$Y_i = \lim(1-\sigma^{12})^{-1}\ \partial_i X = \begin{cases} \xi_2, & i = 1, \\ 0\ , & i \neq 1. \end{cases} \tag{4.2}$$

Definition 4.1. An elementary constant is a linear combination of re-orderings (permutations of the order of the factors) of a fixed monomial.

A constant $C \in B_n$ also implies a concept of closed one-forms.

Proposition 4. If $C \in B_n$,

$$C = C^{i_1 \dots i_n}\xi_{i_1}\dots\xi_{i_n}\ , \tag{4.3}$$

is a constant, then the differential operator

$$\Phi(C)\ :=\ C^{i_1\dots i_n}\partial_{i_1}\dots\partial_{i_n} \tag{4.4}$$

vanishes on B.

Proof. A constant in B_n is a sum of elementary constants; it is enough to prove the proposition for the case that C is an elementary constant. This implies that there are non-zero $f_i \in \mathbb{C}$, $i \in N$, such that the following operator identity

$$\partial_i C - f_i C \partial_i = 0,\ i \in N, \tag{4.5}$$

holds on B. Let B^* be the unital, associative algebra freely generated by $\{\partial_i\}$ $i \in N$, and let $\Phi : B \to B^*$ be the unique isomorphism of algebras such that $\Phi(\xi_i) = \partial_i$. Let $BB^*(q)$ be the unital, associative algebra generated by $\{\xi_i, \partial_i\}$ $i \in N$, with relations (3.3); then Φ extends to a unique isomorphism

$$\Phi' : BB^*(q) \to BB^*(\hat{q}) , \quad \hat{q}^{ij} = 1/q^{ji} .$$

In particular, $\Phi'(\xi_i) = \partial_i$ and $\Phi'(\partial_i) = -(q^{ii})^{-1}\xi_i$, $i \in N$. Now Eq.(4.5) means that $\partial_i \circ C = C f_i \circ \partial_i$, where $a \circ b$ denotes the product in $BB^*(q)$. Applying Φ' one gets

$$\Phi(C) \circ \xi_i = (f_i)^{-1} \xi_i \circ \Phi(C) ,$$

implying that $\Phi(C)X = 0$, $X \in B$.

Definition 4.2. Let C be an elementary constant in B_n, $n \geq 2$. A B_1^* one-form Y, valued in B, will be said to be C-closed if

$$d_C Y \ := \ C^{i_1 \ldots i_n} \partial_{i_1} \ldots \partial_{i_{n-1}} Y_{i_n} = 0 .$$

Examples. In B_2 the constants are of the type $C = \xi_1\xi_1$ or $C' = \xi_1\xi_2 - q^{21}\xi_2\xi_1$. Now Y is C-closed if $dY \ := \ \partial_1 Y_1 = 0$ and C'-closed if $d'Y \ := \ \partial_1 Y_2 - q^{21}\partial_2 Y_1 = 0$. The first case is characteristic of Grassmann algebras and the other of quantum planes. Let $\mathcal{C}$ be the collection

$$\{C_{ij} = \xi_i\xi_j - q^{ji}\xi_j\xi_i , \quad i, j \in N , \quad i \neq j \},$$

and suppose all of them constant. Then we say that a one-form Y is $\mathcal{C}$-closed if Y is C_{ij}-closed for all $i \neq j$:

$$\partial_i Y_j - q^{ji}\partial_j Y_i = 0 , \quad i, j \in N , \quad i \neq j .$$

In this case the closure of a B_1^* one-form is expressed in terms of the B_1^* two-form

$$Z = dY , \quad Z_{ij} = \partial_i Y_j - q^{ji}\partial_j Y_i ,$$

and this naturally leads to familiar q-deformed de Rham complexes, with trivial cohomology.

5. Integrability of Eq. (2.14).

It was seen, in Section 2, that a necessary condition for the standard R-matrix (2.5) to satisfy the Yang-Baxter relation (2.6) is that the coefficients $t_{(\alpha)}^{(\alpha')}$ satisfy (2.14); namely

$$
\begin{aligned}
&[t_{\alpha_1\dots\alpha_\ell}, e_{-\gamma}] = e^{\varphi(\gamma,\cdot)}\delta^\gamma_{\alpha_1} t_{\alpha_2\dots\alpha_\ell} - t_{\alpha_1\dots\alpha_{\ell-1}}\delta^\gamma_{\alpha_\ell} e^{-\varphi(\cdot,\gamma)} \ , \\
&t_{\alpha_1\dots\alpha_\ell} \ := \ t^{\alpha'_1\dots\alpha'_\ell}_{\alpha_1\dots\alpha_\ell} e_{\alpha'_1}\dots e_{\alpha'_\ell} \ .
\end{aligned}
\tag{5.1}
$$

Define * two differential operators, $\vec{\partial}_{-\gamma}$ and $\overleftarrow{\partial}_{-\gamma}$, on $\mathcal{A}^+$, by

$$
[X, e_{-\gamma}] = e^{\varphi(\gamma,\cdot)}\vec{\partial}_{-\gamma} X - X\overleftarrow{\partial}_{-\gamma} e^{-\varphi(\cdot,\gamma)} \ , \tag{5.2}
$$

$X \in \mathcal{A}^+$; note that $\overleftarrow{\partial}_{-\gamma}$ operates from the right. Similarly,

$$
[e_\alpha, Y] = Y\overleftarrow{\partial}_\alpha e^{\varphi(\alpha,\cdot)} - e^{-\varphi(\cdot,\alpha)}\vec{\partial}_\alpha Y \tag{5.3}
$$

defines two differential operators on $\mathcal{A}^-$. These definitions are equivalent to the rules

$$
\begin{aligned}
\vec{\partial}_{-\gamma} e_\alpha &= \delta^\gamma_\alpha + e^{-\varphi(\gamma,\alpha)} e_\alpha \vec{\partial}_{-\gamma} \ , \\
e_\alpha \overleftarrow{\partial}_{-\gamma} &= \delta^\gamma_\alpha + e^{-\varphi(\alpha,\gamma)} \overleftarrow{\partial}_{-\gamma} e_\alpha \ , \\
e_{-\alpha} \overleftarrow{\partial}_{\gamma} &= \delta^\gamma_\alpha + e^{-\varphi(\gamma,\alpha)} \overleftarrow{\partial}_{\gamma} e_{-\alpha} \ , \\
\vec{\partial}_{\gamma} e_{-\alpha} &= \delta^\gamma_\alpha + e^{-\varphi(\alpha,\gamma)} e_{-\alpha} \vec{\partial}_{\gamma} \ .
\end{aligned}
\tag{5.4}
$$

Eq. (5.1) is equivalent to*

$$
\vec{\partial}_{-\gamma} t_{\alpha_1\dots\alpha_\ell} = \delta^\gamma_{\alpha_1} t_{\alpha_2\dots\alpha_\ell} \ , \quad t_{\alpha_1\dots\alpha_\ell}\overleftarrow{\partial}_{-\gamma} = t_{\alpha_1\dots\alpha_{\ell-1}}\delta^\gamma_{\alpha_\ell} \ . \tag{5.5}
$$

Proposition 5. Suppose that the parameters $\varphi(\alpha,\beta)$ are in general position, so that there are no constants in $\mathcal{A}^+$ ($\mathcal{A}^-$) with respect to the differential operators $\vec{\partial}_{-\gamma}$ or $\overleftarrow{\partial}_{-\gamma}$ ($\vec{\partial}_\gamma$ or $\overleftarrow{\partial}_\gamma$). Then either one of the two equations in (5.5) determines $t_{\alpha_1\dots\alpha_\ell}$ recursively and uniquely (the same in each case) from $t_\alpha = e_\alpha$.

Proof. From (5.4) we deduce that

$$
(\vec{\partial}_{-\gamma} X)\overleftarrow{\partial}_{-\gamma'} = \vec{\partial}_{-\gamma}(X\overleftarrow{\partial}_{-\gamma'}) \ . \tag{5.6}
$$

---

* This is where we need the last condition in Definition 2.1.

By Proposition 3, the first of (5.5) determines $t_\ell = t_{\alpha_1\ldots\alpha_\ell}$ uniquely from $t_\alpha = e_\alpha$. The other recursion relation also has a unique solution, t'_ℓ say. We must show that $t_\ell = t'_\ell,\ \ell > 1$. Since parentheses are superfluous,

$$\vec{\partial}_{-\gamma}\, t_\ell \overleftarrow{\partial}_{-\gamma'} = \delta^\gamma_{\alpha_1}\, t_{\ell-1} \overleftarrow{\partial}_{-\gamma'}\ ,$$
$$\vec{\partial}_{-\gamma}\, t'_\ell \overleftarrow{\partial}_{-\gamma'} = \vec{\partial}_{-\gamma}\, t'_{\ell-1}\, \delta^{\gamma'}_{\alpha_\ell}\ .$$

Suppose $t'_k = t_k$ for $k = 1, \ldots, \ell-1$; then the right-hand sides are both equal to $\delta^\gamma_{\alpha_1}\, t_{\ell-2}\, \delta^{\gamma'}_{\alpha_\ell}$. Then the left-hand sides are also equal and, since there are no constants, $t_\ell = t'_\ell$. Since $t_1 = t'_1$ ($t_\alpha = t'_\alpha = e_\alpha$), the proposition follows by induction.

We also encounter the relation

$$\begin{aligned} [e_\alpha, t^{\gamma_1\cdots\gamma_n}] &= t^{\gamma_1\cdots\gamma_{n-1}}\delta^{\gamma_n}_\alpha e^{\varphi(\alpha,\cdot)} - e^{-\varphi(\cdot,\alpha)}\delta^{\gamma_1}_\alpha t^{\gamma_2\cdots\gamma_n}\ , \\ t^{\gamma_1\cdots\gamma_n}\ &:=\ t^{\gamma_1\cdots\gamma_n}_{\gamma'_1\cdots\gamma'_n} e_{-\gamma'_1}\ldots e_{-\gamma'_n}\ . \end{aligned} \tag{5.7}$$

It is equivalent to (5.1) and to either of the following:

$$t^{\gamma_1\cdots\gamma_n}\overleftarrow{\partial}_\alpha = t^{\gamma_1\cdots\gamma_{n-1}}\delta^{\gamma_n}_\alpha\ ,\quad \vec{\partial}_\alpha t^{\gamma_1\cdots\gamma_n} = \delta^{\gamma_1}_\alpha t^{\gamma_2\cdots\gamma_n}\ . \tag{5.8}$$

The proof is similar to that of Proposition 5.

6. Completion of the Proof of Theorem 2.

Suppose that the relations (2.14) are satisfied for $\ell \geq 1$. Now fix $\ell, n,\ \alpha_1, \ldots, \alpha_\ell$ and $\gamma^1, \ldots, \gamma^n$; we must prove that the expression (2.15), summed over m from 0 to min (ℓ, n), vanishes.

We begin by calculating the sum over $m = 0, 1$ (step 1); then we postulate a formula for the partial sum over $m = 0, \ldots, k$ (step k). We prove the formula by induction in k, and finally show that the sum vanishes when $k = \min\,(\ell, n)$.

The term $m = 0$ in (2.15) is

$$[t_\ell, t^n] = t^{\gamma_1\cdots\gamma_n}_{\gamma'_1\cdots\gamma'_n}\sum_{i=1}^n e_{-\gamma'_1}\ldots e_{-\gamma'_{i-1}}[t_\ell, e_{-\gamma'_i}]e_{-\gamma'_{i+1}}\ldots e_{-\gamma'_n}\ . \tag{6.1}$$

As in the preceding section we often write t_ℓ, t^n for $t_{\alpha_1\ldots\alpha_\ell}$, $t^{\gamma_1\cdots\gamma_n}$. We shall gradually make the formulas more schematic so as to bring out their structure. By (2.14)

$$= t^{(\gamma)}_{(\gamma')} \sum_{i=1}^{n} e_{-\gamma'_1} \ldots \left(e^{\varphi(\gamma'_i,\cdot)} \delta^{\gamma'_i}_{\alpha_1} t_{\ell-1} - t_{\ell-1} \delta^{\gamma'_i}_{\alpha_\ell} e^{-\varphi(\cdot,\gamma'_i)} \right) \ldots e_{-\gamma'_n} \ . \tag{6.2}$$

The term $m = 1$ is

$$\begin{aligned} &t^{\gamma_1}_{\alpha_\ell} t_{\alpha_1\ldots\alpha_{\ell-1}} e^{-\varphi(\cdot,\gamma_1)} t^{\gamma_2\ldots\gamma_n} - t^{\gamma_n}_{\alpha_1} t^{\gamma_1\ldots\gamma_{n-1}} e^{\varphi(\alpha_1,\cdot)} t_{\alpha_2\ldots\alpha_\ell} \\ &\quad = t_{\ell-1} e^{-\varphi(\cdot,\alpha_\ell)} \vec{\partial}_{\alpha_l} t^{\gamma_1\cdots\gamma_n} - t^{\gamma_1\cdots\gamma_n} \overleftarrow{\partial}_{\alpha_1} e^{\varphi(\alpha_1,\cdot)} t_{\ell-1} \\ &= t_{\ell-1} t^{(\gamma)}_{(\gamma')} \sum e_{-\gamma'_1} \ldots \delta^{\gamma'_i}_{\alpha_\ell} e^{-\varphi(\cdot,\gamma'_i)} \ldots e_{-\gamma'_n} - t^{(\gamma)}_{(\gamma')} \sum e_{-\gamma'_1} \ldots \delta^{\gamma'_i}_{\alpha_1} e^{\varphi(\gamma'_i,\cdot)} \ldots e_{-\gamma'_n} t_{\ell-1} \ . \end{aligned}$$

This agrees with (6.2) except for the position of $t_{\ell-1}$, and the sign. Thus, adding the contributions $m = 0, 1$ we obtain

$$\begin{aligned} t^{(\gamma)}_{(\gamma')} \sum_{i<j} \big\{ & e_{-\gamma'_1} \ldots \delta^{\gamma'_i}_{\alpha_1} e^{\varphi(\gamma'_i,\cdot)} e_{-\gamma'_{i+1}} \ldots [t_{\ell-1}, e_{-\gamma'_j}] \ldots e_{-\gamma'_n} \\ & + e_{-\gamma'_1} \ldots [t_{\ell-1}, e_{-\gamma'_i}] e_{-\gamma'_{i+1}} \ldots \delta^{\gamma'_j}_{\alpha_l} e^{-\varphi(\cdot,\gamma'_j)} \ldots e_{-\gamma'_n} \big\} \ . \end{aligned} \tag{6.3}$$

This completes step 1; all terms involving t_ℓ have disappeared and $t_{\ell-1}$ appears only in commutators that allow us to use (2.14) again.

We claim that after carrying out step k, which includes summing over $m = 0, \ldots, k$, one obtains the following expression

$$\sum_{s=0}^{k} \ldots (\delta e^{\varphi})^{k-s} \ldots [t_{\ell-k}, e_{-\gamma}] \ldots (\delta e^{-\varphi})^s \ldots \ , \ k < \min(l, n), \tag{6.4}$$

and zero, $k = \min(l, n)$. Here the dots stand for products of the $e_{-\gamma'_i}$, interrupted $k - s$ times by a factor of the type $\delta^{\gamma'_i}_{\alpha_1} e^{\varphi(\gamma'_i,\cdot)}$, once by $[\ ,\ ]$ and s times by a factor like $\delta^{\gamma_i}_{\alpha_l} e^{-\varphi(\cdot,\gamma'_i)}$, as in (6.3).

To verify this claim we carry out the next step. We first evaluate the commutators and examine the cancellations that take place between successive values of s:

$$\begin{aligned} &\ldots [t_{\ell-k}, e_{-\gamma'_i}] \ldots (\delta e^{-\varphi}) \ldots \\ &\quad + \ldots (\delta e^{\varphi}) \ldots [t_{\ell-k}, e_{-\gamma'_i}] \ldots \\ &\qquad = \ldots (\delta^{\gamma'_i}_{\alpha} e^{\varphi(\alpha,\cdot)} t_{\ell-k-1} - t_{\ell-k-1} \delta^{\gamma'_i}_{\alpha'} e^{-\varphi(\cdot,\alpha')}) \ldots (\delta e^{-\varphi}) \ldots \\ &\qquad\quad + (\delta e^{\varphi}) \ldots (\delta^{\gamma'_j}_{\alpha} e^{\varphi(\alpha,\cdot)} t_{\ell-k-1} - t_{\ell-k-1} \delta^{\gamma'_j}_{\alpha'} e^{-\varphi(\cdot,\alpha')}) \ldots \ . \end{aligned}$$

The first term in the first line combines with the second term in the second line to

$$\dots(\delta e^{\varphi})\dots[t_{\ell-k-1}, e_{-\gamma}]\dots(\delta e^{-\varphi})\dots\ .$$

Successive terms in (6.4) all combine in this way, to reproduce the same expression with k replaced by $k+1$, except for the fact that there is no term in the sequence that precedes and collaborates with the first term and no term that succeeds and collaborates with the last term. It remains, therefore, to be proved that the summand $m = k+1$ in (2.15) precisely supplies the two missing terms.

By (5.8),

$$\begin{aligned}
&\vec{\partial}_\beta t^{\gamma_1\cdots\gamma_n} = \delta_\beta^{\gamma_1} t^{\gamma_2\cdots\gamma_n}\ ,\\
&\vec{\partial}_{\beta_m}\dots\vec{\partial}_{\beta_1} t^{\gamma_1\cdots\gamma_n} = \delta_{\beta_1}^{\gamma_1}\dots\delta_{\beta_m}^{\gamma_m}\, t^{\gamma_{m+1}\cdots\gamma_n}\ ,\\
&t^{\beta_1\dots\beta_m}_{\alpha_{\ell-m+1}\dots\alpha_\ell}\vec{\partial}_{\beta_m}\dots\vec{\partial}_{\beta_1} t^{\gamma_1\cdots\gamma_n} = t^{\gamma_1\cdots\gamma_m}_{\alpha_{\ell-m+1}\dots\alpha_l} t^{\gamma_{m+1}\cdots\gamma_n}\ .
\end{aligned}$$

Hence, if $\vec{t}_{\alpha_1\dots\alpha_\ell}$ is the differential operator

$$\vec{t}_{\alpha_1\dots\alpha_\ell}\ :=\ t^{\alpha_1'\dots\alpha_\ell'}_{\alpha_1\dots\alpha_\ell}\vec{\partial}_{\alpha_l'}\dots\vec{\partial}_{\alpha_1'}\ ,$$

then

$$t^{\gamma_1\cdots\gamma_m}_{\alpha_{\ell-m+1}\dots\alpha_\ell} t^{\gamma_{m+1}\cdots\gamma_n} = \vec{t}_{\alpha_\ell\dots\alpha_{\ell-m+1}} t^{\gamma_1\cdots\gamma_n} \tag{6.5}$$

and the first of the two terms in (2.15) is

$$\begin{aligned}
&t_{\alpha_1\dots\alpha_{\ell-m}} e^{-\varphi(\cdot,\sigma)}\vec{t}_{\alpha_\ell\dots\alpha_{\ell-m+1}} t^{\gamma_1\cdots\gamma_n}\\
&\qquad = t_{\alpha_1\dots\alpha_{\ell-m}} t^{(\gamma)}_{(\gamma')}\sum_{i=1}^n e_{-\gamma_1'}\dots\left[e^{-\varphi(\cdot,\sigma)}\vec{t}_{\alpha_\ell\dots\alpha_{\ell-m+1}}, e_{-\gamma_i'}\right]\dots\\
&\qquad = t_{\alpha_1\dots\alpha_{\ell-m}} t^{(\gamma)}_{(\gamma')}\sum_i e_{-\gamma_1'}\dots\delta^{\gamma_i'}_{\alpha_{\ell-m+1}} e^{-\varphi(\cdot,\sigma)}\vec{t}_{\alpha_\ell\dots\alpha_{\ell-m+2}}\dots\ .
\end{aligned}$$

By iteration of these steps one finally ends up, when $m = k+1$, with precisely the missing terms; actually one of the missing terms, we leave it to the reader to carry out the calculation for the other one. This done, the proof of Theorem 2 is complete.

Corollary 5. With the parameters in general position, there exists a unique R-matrix on $\mathcal{A}$ that satisfies the Yang-Baxter relation.

7. Obstructions and Generalized Serre Relations.

We have been concerned with the construction of a standard R-matrix, Definition (2.2), that satisfies the Yang-Baxter relation, Eq.(2.6), in terms of the generators of an algebra $\mathcal{A}$, Definition (2.1). The relations of $\mathcal{A}$ involve parameters; at certain hypersurfaces in parameter space we have encountered obstructions, characterized by the vanishing of one or more of the determinants that we have studied in Section 3. At these points there appear elements in $\mathcal{A}^+$ that are constants with respect to differential operators $\vec{\partial}_{-\alpha}$ and/or $\overleftarrow{\partial}_{-\alpha}$, and elements in $\mathcal{A}^-$ that are constants with respect to $\vec{\partial}_\alpha$ and/or $\overleftarrow{\partial}_\alpha$.

We shall show that all these obstructions can be overcome by the introduction of additional relations in the definition of $\mathcal{A}$ or, what is the same, by replacing $\mathcal{A}$ by a quotient $\mathcal{A}/I$, where I is an ideal generated by the constants. The next three propositions relate the null-spaces of the four differential operators to each other.

Proposition 7.1. The space of constants with respect to $\vec{\partial}_{-\gamma}$ in $\mathcal{A}_n^+$ has the same dimension as the space of constants with respect to $\overleftarrow{\partial}_{-\gamma}$. If there are no constants in $\mathcal{A}_\ell^+$ for $\ell = 1, \ldots, n-1$, then the two spaces coincide.

Proof. An easy consequence of Eq. (5.6).

Let $C \in \mathcal{A}_n^+$ be a constant with respect to $\vec{\partial}_{-\gamma}$, $\gamma \in N$, and assume, provisionally, that there are no constants in $\mathcal{A}_\ell^+$, $1 < \ell < n$. Without essential loss of generality we take C to be a linear combination

$$C = C^{\gamma_1 \ldots \gamma_n} e_{\gamma_1} \ldots e_{\gamma_n} \ ,$$

where the summation runs over the permutations of a fixed set $\{\gamma_1 \ldots \gamma_n\}$, hence over a finite set. Let d be the operator that takes $X \in \mathcal{A}_n^+$ to the one-form Y valued in $\mathcal{A}_{n-1}^+$ with components $\vec{\partial}_{-\gamma} X$, $\gamma \in N$. This operator is represented by a direct sum of finite square matrices, also denoted d. The constant C is a null-vector for d. The transposed matrix also has a null-vector; it exists by virtue of the fact that dX is C-closed: (Definition 4.2):

$$\begin{aligned}
&(C^{\gamma_1 \ldots \gamma_n} \vec{\partial}_{-\gamma_1} \ldots \vec{\partial}_{-\gamma_n}) e_{\alpha_1} \ldots e_{\alpha_n} \\
&\qquad = (C^{\gamma_1 \ldots \gamma_n} \vec{\partial}_{-\gamma_1} \ldots \vec{\partial}_{-\gamma_{n-1}})(d^{\gamma_n \beta_1 \ldots \beta_{n-1}}_{\alpha_1 \ldots \alpha_n} e_{\beta_1} \ldots e_{\beta_{n-1}}) \\
&\qquad = d^{\gamma_n \beta_1 \ldots \beta_{n-1}}_{\alpha_1 \ldots \alpha_n} (C^{\gamma_1 \ldots \gamma_n} \vec{\partial}_{-\gamma_1} \ldots \vec{\partial}_{-\gamma_{n-1}} e_{\beta_1} \ldots e_{\beta_{n-1}}) = 0 \ .
\end{aligned}$$

The obstruction to solving Eq. (5.5) is that the right-hand side is not in the null-space of the transpose of d; it is not C-closed. Indeed, since there are no constants in $\mathcal{A}^+_\ell,\ \ell < n$,

$$(C^{\gamma_1\ldots\gamma_n}\vec{\partial}_{\gamma_1}\ldots\vec{\partial}_{\gamma_{n-1}})\delta^{\alpha_1}_{\gamma_n}t_{\alpha_2\ldots\alpha_n} = C^{\alpha_n\ldots\alpha_1} \neq 0\ .$$

Recall that the R-matrix is expressed in terms of $e_{-\alpha_1}\ldots e_{-\alpha_n}t_{\alpha_1\ldots\alpha_n}$. The obstruction to Yang-Baxter is thus

$$e_{-\alpha_1}\ldots e_{-\alpha_n}C^{\alpha_n\ldots\alpha_1}\ =:\ C' \in \mathcal{A}^-_n\ .$$

Proposition 7.2. The element $C' \in \mathcal{A}^-_n$ is a constant.

Proof. One verifies directly that $\vec{\partial}_{-\gamma}C = 0,\ \gamma \in N$, is equivalent to $C'\overleftarrow{\partial}_\gamma = 0,\ \gamma \in N$.

Thus, if the first obstruction to the Yang-Baxter relation is encountered at the evaluation of $t_{\alpha_1\ldots\alpha_n}$, then this obstruction can be avoided by replacing $\mathcal{A}$ by the quotient $\mathcal{A}/I_n$, where I_n is the ideal generated by the constants in $\mathcal{A}^\pm_n$. Once this has been done, we study the obstructions at the next level. Since the constants at level n have been removed there are none in $\mathcal{A}^\pm_\ell,\ \ell \leq n$, and substantially the same analysis applies to constants in $\mathcal{A}^\pm_{n+1}$. To formulate the final result we need:

Proposition 7.3. The ideal $I^+ \subset \mathcal{A}^+$ generated by the constants of $\vec{\partial}_{-\gamma}$, coincides with that generated by the constants of $\overleftarrow{\partial}_{-\gamma}$. The same statement holds true in $\mathcal{A}^-$, *mutatis mutandis.*

When the parameters are in general position there are no constants, and Theorem 2 with Proposition 5 assures us that there is a unique standard R-matrix in $\mathcal{A}\otimes\mathcal{A}$ that satisfies the Yang-Baxter relation (2.6). We are now in a position to allow for the appearance of constants.

Remark. There are no constants in $\mathcal{A}^\pm_1$; the generators $H_a, e_{\pm\alpha}$ of $\mathcal{A}$ are also generators of $\mathcal{A}' = \mathcal{A}/I$.

Theorem 7. Let $I \subset \mathcal{A}$ be the ideal generated by the constants in $\mathcal{A}^+$ and the constants in $\mathcal{A}^-$, and let $\mathcal{A}'$ be the quotient $\mathcal{A}/I$. Interpret the standard, universal R-matrix (2.5)

as an element of $\mathcal{A}' \otimes \mathcal{A}'$. The Yang-Baxter relation for the standard R-matrix on $\mathcal{A}'$ is equivalent to the recursion relation

$$\begin{aligned} [t_l, e_{-\gamma}] &= (e_{-\gamma} \otimes e^{\varphi(\gamma,\cdot)})t_{l-1} - t_{l-1}(e_{-\gamma} \otimes e^{-\varphi(\cdot,\gamma)}), \\ t_l &:= t^{(\alpha')}_{(\alpha)} e_{-\alpha_1} \dots e_{-\alpha_l} \otimes e_{\alpha'_1} \dots e_{\alpha'_l}, \end{aligned} \tag{7.1}$$

and to either one of the following

$$[e_\gamma, t_l] = t_{l-1}(e^{\varphi(\gamma,\cdot)} \otimes e_\gamma) - (e^{-\varphi(\cdot,\gamma)} \otimes e_\gamma)t_{l-1}, \tag{7.2}$$

$$(1 \otimes \vec{\partial}_{-\gamma})t_l = (e_{-\gamma} \otimes 1)t_{l-1}, \quad t_l(1 \otimes \overleftarrow{\partial}_{-\gamma}) = t_{l-1}(e_{-\gamma} \otimes 1), \tag{7.3}$$

$$t_l(\overleftarrow{\partial}_\gamma \otimes 1) = t_{l-1}(1 \otimes e_\gamma), \quad (\vec{\partial}_\gamma \otimes 1)t_l = (1 \otimes e_\gamma)t_{l-1}. \tag{7.4}$$

These relations are integrable (with $t_1 = e_{-\alpha} \otimes e_\alpha$) and yield a unique standard R-matrix on $\mathcal{A}'$.

8. Deformations.

This work was initiated with the aim of calculating the universal R-matrices associated with simple Lie algebras, as deformations of the standard universal R-matrix. We shall establish a direct correspondence between the classical r-matrices of Belavin and Drinfeld on the one hand, and the deformations of the standard, universal R-matrix for simple quantum groups on the other. In preparation for this we have explored the meaning of the Yang-Baxter relation in a much more general context, and we shall endeavor to maintain this generality in our approach to deformations. But, as for the types of deformations, we shall limit our study in a way that seems natural in the context of quantum groups.

A deformation of the standard R-matrix is a formal series

$$R_\epsilon = R + \epsilon R_1 + \epsilon^2 R_2 + \dots \ . \tag{8.1}$$

Here R is given by Eq. (2.5), with the coefficients determined by the Yang-Baxter relation, and we attempt to find $R_1, R_2, \dots$ so that R_ϵ will satisfy the same relation to each order

in ϵ. To make this program precise, we must specify the nature of the leading term; the remainder should then be more or less unique.

Recall that R "commutes with Cartan." An element $Q \in \mathcal{A} \otimes \mathcal{A}$ is said to have weight w if

$$[H_a \otimes 1 + 1 \otimes H_a,\ Q] = w_a Q\ , \quad w_a \in \mathbb{C},\ a \in M\ . \tag{8.2}$$

Thus R has weight zero. The image of Q by the projection $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}' \otimes \mathcal{A}'$ has the same weight. We shall suppose that R_1 is homogeneous (has weight).

Recall further that R is driven by the linear term; by virtue of the Yang-Baxter relation, R is completely determined by the term $e_{-\alpha} \otimes e_\alpha$. It is natural to study deformations that are driven by a similar term, with fixed, non-zero weight:

$$R_1 = S(e_{\pm\sigma} \otimes e_{\pm\rho}) + \dots\ , \tag{8.3}$$

with σ, ρ fixed and the factor S is in $\mathcal{A}^0$. The unwritten terms are of higher order, in a sense that we must make precise.

Proposition 8. The algebra $\mathcal{A}' = \mathcal{A}/I$ is $\mathbb{Z}$-graded, with grade $e_{\pm\alpha} = \pm 1$, grade $H_a = 0$. An alternative grading is obtained by reversing the sign.

Proof. This is a consequence of the fact that the generators of I are homogeneous; $\mathcal{A}'$ inherits the grading of $\mathcal{A}$.

The standard R-matrix is a formal series $\sum_k \psi_k^- \otimes \psi_k^+,\ \psi_k^\pm \in \mathcal{A}'$. We use the grading of Proposition 8 in the second space, the alternative grading in the first space; then grade $\psi_k^\pm = k$ and R is a formal sum of terms with grade $(k,k),\ k = 0,1,2,\dots$. This grading is an extension of that used previously, made necessary by the appearance of e_σ in the first space and $e_{-\rho}$ in the second.

With the inclusion of (8.3) the grades descend to $(-1,-1)$. Finally, the unwritten terms in (8.3) is a series by ascending grades. The fact that the grades are bounded below is fundamental. We claim that R_ϵ, a formal series in ϵ, each term a formal series in ascending grades, if it satisfies the Yang-Baxter relation, is completely determined by the choice of the two generators $e_{\pm\sigma}$ and $e_{\pm\rho}$ in (8.3).

We shall see that the standard R-matrix on $\mathcal{A}'$, with the parameters of $\mathcal{A}'$ in general position, is rigid with respect to deformations of the type (8.3). We begin our investigation by establishing some conditions on the parameters that are necessary for the existence of a deformation. In the sections that follow we shall study each of the four possibilities envisaged by (8.3) separately. We organize the contributions to

$$YB_\epsilon \ := \ R_{\epsilon 12}R_{\epsilon 13}R_{\epsilon 23} - R_{\epsilon 23}R_{\epsilon 13}R_{\epsilon 12}$$

in the same way as the contributions to YB. A term $\psi_1 \otimes \psi_2 \otimes \psi_3$ is said to have grade (ℓ, n) if ψ_3 has grade n and ψ_1 has alternative grade ℓ. We limit ourselves, in this part, to terms linear in ϵ.

9. Deformations of Types $e_{-\sigma} \otimes e_\rho$, $e_\sigma \otimes e_\rho$ and $e_{-\sigma} \otimes e_{-\rho}$.

We suppose that the driving term in R_1 is

$$S(e_{-\sigma} \otimes e_\rho) \ , \quad S \in \mathcal{A}^0 \otimes \mathcal{A}^0 \ .$$

We examine the contributions to YB_ϵ of order ϵ.

The lowest grades are (1,0) and (0,1), with contributions

$$\begin{aligned}
&(Se_{-\sigma} \otimes e_\rho)_{12} \ R^0_{13}R^0_{23} - R^0_{23}R^0_{13}(Se_{-\sigma} \otimes e_\rho)_{12} \ , \\
&R^0_{12}R^0_{13}(Se_{-\sigma} \otimes e_\rho)_{23} - (Se_{-\sigma} \otimes e_\rho)_{23}R^0_{13}R^0_{12} \ ,
\end{aligned}$$

respectively. These vanish if and only if

$$e^{\varphi(\sigma,\cdot) - \varphi(\rho,\cdot)} = 1 = e^{\varphi(\cdot,\sigma) - \varphi(\cdot,\rho)} \ . \tag{9.1}$$

In grade (1,1) we have three contributions, the simplest one is

$$\begin{aligned}
&A = F\big(e_{-\sigma} \otimes \{e^{-\varphi(\cdot,\rho)} - e^{\varphi(\sigma,\cdot)}\} \otimes e_\rho\big) \ , \\
&F = R^0_{12}S_{13}R^0_{23} \ .
\end{aligned}$$

The other contributions to the same grade are, in view of (9.1),

$$
\begin{aligned}
B = {} & R^i R^j e_{-\alpha} \otimes R_i e_\alpha S^k e_{-\sigma} \otimes R_j e^{\varphi(\alpha,\cdot)} S_k e_\rho \\
& - R^i R^j e_{-\alpha} \otimes R_i S^k e_\alpha e_{-\sigma} \otimes R_j S_k e_\rho \ ,
\end{aligned}
$$

and

$$
\begin{aligned}
C = {} & S^i R^j e_{-\sigma} \otimes R^k S_j e_\rho e_{-\alpha} \otimes R_j R_k e_\alpha \\
& - S^i R^j e^{-\varphi(\cdot,\alpha)} e_{-\sigma} \otimes R^k e_{-\alpha} S_j e_\rho \otimes R_j R_k e_\alpha \ .
\end{aligned}
$$

No cancellations occur unless

$$
S = R^0 \ ; \tag{9.2}
$$

the form of the Cartan factor S is already fixed. Now

$$
\begin{aligned}
B &= F\big(e_{-\sigma} \otimes \{e^{\varphi(\sigma,\cdot)} - e^{-\varphi(\cdot,\sigma)}\} \otimes e_\rho\big) \ , \\
C &= F\big(e_{-\sigma} \otimes \{e^{\varphi(\rho,\cdot)} - e^{-\varphi(\cdot,\rho)}\} \otimes e_\rho\big) \ .
\end{aligned}
$$

The sum $A + B + C$ vanishes if and only if

$$
e^{\varphi(\rho,\cdot)+\varphi(\cdot,\sigma)} = 1 \ . \tag{9.3}
$$

Let us take stock. Conditions (9.1)-(9.3) are necessary. From (9.1) and (9.3) it follows in particular that

$$
e^{\varphi(\rho,\alpha)+\varphi(\alpha,\rho)} = 1 = e^{\varphi(\sigma,\alpha)+\varphi(\alpha,\sigma)} \ , \quad \alpha \in N \ . \tag{9.4}
$$

These are conditions that are familiar from our investigation of constants, see Eq.(3.8). The relations that are thus implied are

$$
e_\rho e_\alpha - e^{\varphi(\rho,\alpha)} e_\alpha e_\rho = 0 = e_\sigma e_\alpha - e^{\varphi(\sigma,\alpha)} e_\alpha e_\sigma \ , \ \forall \alpha \in N.
$$

This constitutes a high degree of commutativity in $\mathcal{A}'$ and takes us far away from our main interest in simple quantum groups. We therefore end our investigation of the type $e_{-\sigma} \otimes e_\rho$ at this point.

Suppose next that the driving term in R_1 is

$$
S(e_\sigma \otimes e_\rho) \ , \quad S \in \mathcal{A}^0 \otimes \mathcal{A}^0 \ .
$$

The lowest grade (in YB_ϵ) is (-1,0), with just one contribution,

$$F'\big(e_\sigma \otimes e_\rho \otimes \{e^{-\varphi(\sigma,\cdot)-\varphi(\rho,\cdot)} - 1\}\big) \ ,$$
$$F' \ := \ S_{12}R^0_{13}R^0_{23} \ ;$$

so we need

$$e^{\varphi(\sigma,\cdot)+\varphi(\rho,\cdot)} = 1 \ . \tag{9.5}$$

The next lowest grade is (-1,1), with two contributions. The simpler one is

$$A_1 = F\big(e_\sigma \otimes \{e^{-\varphi(\cdot,\rho)} - e^{-\varphi(\sigma,\cdot)}\} \otimes e_\rho\big) \ ,$$

and the other one

$$B_1 = S^i R^j e_\sigma \otimes S_i R^k e_\rho e_{-\alpha} \otimes R_j R_k e_\alpha$$
$$- S^i R^j e^{-\varphi(\cdot,\alpha)} e_\sigma \otimes R^k e_{-\alpha} S_i e_\rho \otimes R_j R_k e_\alpha \ .$$

If the two terms in B_1 are to combine to something that can be cancelled by A_1 we need $S = (KR^i) \otimes R_i, \ K \in \mathcal{A}^0$, in which case

$$B_1 = F\big(Ke_\sigma \otimes \{e^{\varphi(\rho,\cdot)} - e^{-\varphi(\cdot,\rho)}\} \otimes e_\rho\big) \ .$$

We need to make $A_1 + B_1 = 0$. In view of (9.5) it can be accomplished in two ways:

$$\text{(i)} \ \ K = 1 \ , \quad \text{or} \quad \text{(ii)} \ \ e^{\varphi(\rho,\cdot)+\varphi(\cdot,\rho)} = 1 \ .$$

The second solution is very restrictive; it implies that e_ρ quommutes with all the $e_\alpha, \ \alpha \in N$; besides, it is excluded by the last condition of Definition 2.1.

We therefore abandon (ii) and adopt (i); that is,

$$S = R^0 \ .$$

The next lowest grade is (0,1), with three contributions. Using (9.5) they can be reduced to

$$A_2 = F\big(\{1 - e^{-\varphi(\cdot,\sigma)-\varphi(\cdot,\rho)}\} \otimes e_\sigma \otimes e_\rho\big) \ ,$$
$$B_2 = F\big(e_\sigma e_{-\alpha} \otimes e^{\varphi(\rho,\alpha)-\varphi(\cdot,\alpha)} e_\rho \otimes e_\alpha$$
$$- e_{-\alpha} e_\sigma \otimes e^{\varphi(\alpha,\cdot)} e_\rho \otimes e_\alpha\big)$$
$$C_2 = F\big(e_{-\alpha} e_\sigma \otimes e^{\varphi(\alpha,\rho)-\varphi(\cdot,\rho)} e_\alpha \otimes e_\rho$$
$$- e_\sigma e_{-\alpha} \otimes e^{-\varphi(\sigma,\cdot)} e_\alpha \otimes e_\rho\big) \ .$$

Cancellation requires very strong conditions on the parameters and we shall stop at this point. Similar results are obtained for deformations of type $e_{-\sigma} \otimes e_{-\rho}$.

10. Deformations of Type $e_\sigma \otimes e_{-\rho}$.

We come to the last case envisaged in Section 8. Eq. (8.3), when the driving term in R_1 has the form

$$S(e_\sigma \otimes e_{-\rho}) \ , \quad S \in \mathcal{A}^0 \otimes \mathcal{A}^0 \ . \tag{10.1}$$

This term has grade (-1,-1); it is the only term in R_1 with this grade, the lowest. The factor S, and all other terms in R_1, are completely determined by the Yang-Baxter relation $YB_\epsilon = 0$ to first order in ϵ. Besides (10.1) there is in R_1 one other term with only two roots, of the form

$$S'(e_{-\rho} \otimes e_\sigma) \ , S' \in \mathcal{A}^0 \otimes \mathcal{A}^0 \ ; \tag{10.2}$$

it has grade (1,1).

Theorem 10. Let R be the standard R-matrix described in Theorem 7. Suppose that $R + \epsilon R_1$ is a first order deformation, satisfying the Yang-Baxter relation to first order in ϵ. Suppose finally that the term of lowest grade in R_1 has the form (10.1); then the parameters satisfy

$$e^{\varphi(\cdot,\rho)+\varphi(\sigma,\cdot)} = 1 \ ; \tag{10.3}$$

and R_1 is uniquely determined and has the expression

$$R_1 = R(Ke_\sigma \otimes Ke_{-\rho}) - (Ke_{-\rho} \otimes Ke_\sigma)R \ , \tag{10.4}$$

with $K \ := \ e^{\varphi(\cdot,\rho)}$.

Proof. An easy calculation in the lowest grades shows that (10.3) is necessary and that $S = K \otimes K$, up to a numerical factor that we fix once and for all.

Let $R_1^i, \ i = 1,2$, be the two summands in (10.4). The term of order ϵ in YB_ϵ is the sum of the following six quantities:

$$\begin{aligned} A_{12}^i &= (R_1^i)_{12} R_{13} R_{23} - R_{23} R_{13} (R_1^i)_{12} \ , \\ A_{13}^i &= R_{12} (R_1^i)_{13} R_{23} - R_{23} (R_1^i)_{13} R_{23} \ , \\ A_{23}^i &= R_{12} R_{13} (R_1^i)_{23} - (R^i)_{23} R_{13} R_{12} \ , \quad i = 1,2 \ . \end{aligned} \tag{10.5}$$

Step 1. We begin with the term that contains the lowest grade, (1,1):

$$\begin{aligned} &A_{13}^1 = R^i[-\alpha]R^j[-\beta]Ke_\sigma \otimes R_i[\alpha]R^k[-\gamma] \otimes R_j[\beta]Ke_{-\rho}R_k[\gamma] - \dots \ , \\ &[-\alpha]\otimes[\alpha] \ := \ e_{-\alpha_1}\dots e_{-\alpha_\ell}\, t_{(\alpha)}^{(\alpha')}\, e_{\alpha_1'}\dots e_{\alpha_\ell'} \ . \end{aligned} \tag{10.6}$$

A sum over indices and numbers of indices (ℓ α's, m β's and n γ's) is understood, and $-\dots$ stands for the reflected term. Using the fact that R satisfies $YB = 0$ we can convert (10.6) to

$$A_{13}^1 = R^i[-\alpha]R^j[-\beta]Ke_\sigma \otimes R_i[\alpha]R^kK[-\gamma] \otimes R_j[\beta]KR_k\big[e_{-\rho},[\gamma]\big] \ + \dots, \tag{10.7}$$

where $+\dots$ stands for a similar expression that contains a factor $\big[e_\sigma,[-\alpha]\big]$ in the first space. We have used (10.3) and continue to use this relation without comment.

Step 2. Evaluate the commutators in (10.7) using (5.1) and (5.7). The result

$$A_{13}^1 = R^i[-\alpha]R^j[-\beta]Ke_\sigma \otimes R_i[\alpha]R^kK[-\gamma]e_{-\rho} \otimes R_j[\beta]KR_k[\gamma]K^{-1} + \dots \tag{10.8}$$

is a sum of four similar expressions. Note that the evaluation of the commutators involves a shift in the summation indices ℓ, m, n. The lowest grades in (10.8) are (-1,0) and (0,-1).

Step 3. Now write down the full expression for $A_{12}^1 + A_{23}^1$; it also contains four similar terms. Two of them cancel two of the terms in (10.8); to verify this the relation $YB = 0$ must be invoked.

Step 4. Combine the remaining two terms from (10.8) with the remaining two terms from $A_{12}^1 + A_{23}^1$ and verify that

$$\begin{aligned} &A_{12}^1 + A_{13}^1 + A_{23}^1 \\ &= R^i[-\alpha]KR^j\big[e_\sigma,[-\beta]\big] \otimes R_i[\alpha]KR^ke_{-\rho}[-\gamma] \otimes R_jK[\beta]R_ke^{\varphi(\rho,\cdot)}[\gamma] + \dots \ , \end{aligned} \tag{10.9}$$

where $+\dots$ stands for a term that contains a factor $\big[[\beta],e_{-\rho}\big]$ in the third space.

Step 5. Evaluate the commutators (second shift of summation indices)

$$\begin{aligned} &= R^i[-\alpha]R^jK[-\beta]K^{-1} \otimes R^i[\alpha]KR^ke_{-\rho}[-\gamma] \otimes R_jK[\beta]e_\sigma R_ke^{\varphi(\rho,\cdot)}[\gamma] + \dots \\ &=: \ X_1 + X_2 + Y_1 + Y_2 \ . \end{aligned} \tag{10.10}$$

The lowest grades are now (1,0) and (0,1).

Step 6. Two of the four terms in (10.10) are:

$$\begin{aligned} X_1 &= R_{12}R_{13}\big\{(Ke_{-\rho}\otimes Ke_\sigma)R\big\}_{23}\ , \\ Y_2 &= -R_{23}R_{13}\big\{(Ke_{-\rho}\otimes Ke_\sigma)R\big\}_{12}\ . \end{aligned} \tag{10.11}$$

Now add $A^2_{12} + A^2_{23}$ to (10.10) to get

$$\begin{aligned} &A^1_{12} + A^1_{13} + A^1_{23} + A^2_{12} + A^2_{23} \\ &\quad = \tilde{X}_1 + X_2 + Y_1 + \tilde{Y}_2\ , \end{aligned} \tag{10.12}$$

where $\tilde{X}_1$ and $\tilde{Y}_2$ are obtained from X_1 and Y_2 by adding A^2_{23} and A^2_{12}.

Step 7. Use the relation $YB = 0$ to modify the expressions for $\tilde{X}_1$ and $\tilde{Y}_2$; then notice that the four terms in (10.12) can be combined to two,

$$= R^iK[-\alpha]R^j[-\beta]\otimes R_iK[e_{-\rho},[\alpha]]R^k[-\gamma]\otimes Ke_\sigma R_j[\beta]R_k[\gamma] + \ldots, \tag{10.13}$$

where the other term has a factor $[[-\gamma], e_\sigma]$ in the second space.

Step 8. Evaluate the commutators (third shift of summation indices),

$$= R^iK[-\alpha]e_{-\rho}R^j[-\beta]\otimes R_jK[\alpha]K^{-1}R^k[-\gamma]\otimes Ke_\sigma R_j[\beta]R_k[\gamma] + \ldots. \tag{10.14}$$

This expression has four terms; the lowest grade is (1.1).

Step 9. Two of the four terms in (10.14) cancel each other because $YB = 0$.

Step 10. The remaining two terms add up to $-A^2_{13}$.

This completes the verification of the claim that (10.4) defines a first order deformation of R. To complete the proof of Theorem 10 we must show that this expression (10.4) is unique. This was done by complete mathematical induction. We omit the details but point out that the key to the induction processs is visible in steps 2,5 and 8, where the summation indices are shifted. Theorem 10 is proved.

Let $\mathcal{P}$ be the collection of pairs $(\sigma,\rho) \in N \otimes N$ such that (10.3) holds; each distinct pair defines a first order deformation $R+\epsilon R_1^{\sigma,\rho}$ of R. Because these deformations are only first order they generate a linear space

$$R_1 = \sum_{\sigma,\rho\in\mathcal{P}} C_{\sigma,\rho} R_1^{\sigma,\rho}, \tag{10.15}$$

with coefficients in the field. The dimension of this space of first order deformations is zero for parameters in general position. It remains zero, generically, when the parameters are such that the ideal I generated by the constants is non-empty and R is defined on $\mathcal{A}/I$.

11. Classical R-matrices for Simple Lie Algebras.

We shall now specialize, by stages, until we arrive at simple quantum groups, where a confrontation can be made with the list of r-matrices obtained by Belavin and Drinfeld [BD].

Suppose that $\mathrm{Card}(N) \ := \ \ell < \infty$. Suppose next that the ideal I (generated by the constants of $\mathcal{A}$) is generated by a complete set of Serre relations; that is, for each pair $\alpha,\beta \in N,\ \alpha \neq \beta$, there is a smallest positive integer $k_{\alpha\beta}$ such that there is a relation in $\mathcal{A}/I$ of the form

$$\sum_{m=0}^{k_{\alpha\beta}} Q_m^{(\alpha,\beta)} (e_\alpha)^m e_\beta (e_\alpha)^{k_{\alpha\beta}-m} = 0 \ , \tag{11.1}$$

with coefficients $Q_m^{(\alpha\beta)}$ in the field. The left side, as an element of $\mathcal{A}^+$, is a constant, and the penultimate paragraph of Section 3 applies. In particular, the relation (3.20) becomes

$$e^{\varphi(\alpha,\beta)+\varphi(\beta,\alpha)+(k_{\alpha\beta}-1)\varphi(\alpha,\alpha)} = 1 \ , \tag{11.2}$$

We specialize further by supposing that the exponent vanish,

$$\varphi(\alpha,\beta)+\varphi(\beta,\alpha) = (1-k_{\alpha\beta})\varphi(\alpha,\alpha) \ , \quad \alpha \neq \beta \ . \tag{11.3}$$

The form $(\cdot,\cdot)$ defined by

$$(\alpha,\beta) = \varphi(\alpha,\beta)+\varphi(\beta,\alpha) \tag{11.4}$$

will be called the restricted Killing form, and the ℓ-by-ℓ matrix with components

$$A_{\alpha\beta} \ := \ 1 - k_{\alpha\beta} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \tag{11.5}$$

will be called the generalized Cartan matrix; note that it is symmetrizable. Finally, a suitable restriction on Card(M) brings us to quantum Kac-Moody algebras [K][M].

The semi-classical limit of R is defined by replacing

$$\begin{aligned} &\varphi(\cdot,\cdot) \to \hbar\varphi(\cdot,\cdot) \ , \\ &e_\alpha = cE_\alpha \ , \quad e_{-\alpha} = c' E_{-\alpha} \ , \quad cc' = \hbar \ , \quad \alpha \in N \ , \end{aligned} \tag{11.6}$$

and developing the exponentials to first order in $\hbar$. Then Eq. (2.4) becomes

$$\begin{aligned} [E_\alpha, E_{-\beta}] &= \delta_{\alpha\beta}\big(\varphi(\alpha,\cdot) + \varphi(\cdot,\alpha)\big) \\ &=: \ \delta_{\alpha\beta} H_{(\alpha)}\varphi(\alpha,\alpha) \ . \end{aligned} \tag{11.7}$$

It follows from (11.7) and (2.3) that

$$[H_{(\alpha)}, E_\beta] = A_{\alpha\beta} E_\beta \ , \quad \alpha,\beta \in N \ . \tag{11.8}$$

The definition (11.5) of the generalized Cartan matrix implies that $A_{\alpha\alpha} = 2, \ \alpha \in N$, that $A_{\alpha\beta} \in \{0,-1,-2,\ldots\}, \ \alpha \neq \beta$, and that $A_{\alpha\beta} \neq 0$ implies $A_{\beta\alpha} \neq 0$. Special cases are affine Lie algebras and simple Lie algebras. The latter are characterized by two additional properties of $(A_{\alpha\beta})$: indecomposability and $\det(A) > 0$. We may now assume that both hold, and that $\{H_{(\alpha)}, \, \alpha \in N\}$ generates $\mathcal{A}^0$.

The (classical) r-matrix r, associated with the standard R-matrix (2.5) is defined by

$$R = 1 + \hbar r + o(\hbar^2) \ . \tag{11.9}$$

Two terms in r are obvious: $r = \varphi + \sum E_{-\alpha} \otimes E_\alpha + ?$, with the sum extending over simple roots. Evaluating the remaining terms is more difficult, but the result is known. With a particular normalization of the non-simple roots,

$$r = \varphi + \sum_{\alpha\in\Delta^+} E_{-\alpha} \otimes E_\alpha \ , \tag{11.10}$$

where Δ^+ is the set of positive roots. This may be called the standard r-matrix. It satisfies the classical Yang-Baxter relation

$$[r_{12}, r_{13} + r_{23}] + [r_{13}, r_{23}] = 0 \tag{11.11}$$

and

$$r + r^t = \hat{K} \ , \tag{11.12}$$

the Killing form of $\mathcal{L}$. In the list of (constant) r-matrices obtained by Belavin and Drinfeld [BD], (11.10) is the simplest. The quantum groups to which these r-matrices are associated are the twisted quantum groups of Reshetikhin and others [R][Sc][Su].

To any first order deformation of R, there corresponds a first order deformation of r,

$$R_\epsilon = 1 + \hbar r_\epsilon + o(\hbar^2), \quad r_\epsilon = r + \epsilon r_1 + o(\epsilon^2) \ . \tag{11.13}$$

Eqs. (10.4) and (10.5) give us

$$r_1 = \sum_{\sigma,\rho\in\mathcal{P}} C_{\sigma,\rho}(E_\sigma \wedge E_{-\rho}) \ , \tag{11.14}$$

where $\mathcal{P}$ is the set of pairs with the property

$$\varphi(\rho,\cdot) + \varphi(\cdot,\sigma) \ := \ 0 \ . \tag{11.15}$$

The original work of Belavin and Drinfeld culminates in a list of constant r-matrices that is complete up to equivalence. Their results have recently been re-derived in terms of deformation theory and the associated cohomology.

Proposition 11. [F] Let r be the standard r-matrix (11.10) for a simple Lie algebra $\mathcal{L}$. The space of essential, first order deformations of r, satisfying (11.11) and (11.12), is

$$H^2(\mathcal{L}^*,\mathbb{C}) = \big\{r_1 = \sum_{\sigma,\rho\in\mathcal{P}} C_{\sigma,\rho} E_\sigma \wedge E_{-\rho} + \sum \tilde{C}^{ab} H_a \otimes H_b\big\} \ . \tag{11.16}$$

The exact deformations are of finite order and coincide with the r-matrices of [BD].

The second, Cartan term is not "essential" in the present context; it represents the freedom to vary the parameters. We conclude that

Theorem 11. The first order deformations of the standard R-matrix calculated in Section 10, upon specialization to a simple quantum group, are in one-to-one correspondence, via (11.9), with the essential first order deformations of the associated r-matrix (11.10)-modulo variations of the parameters.

12. Hopf Structure.

It is of some interest to verify that the standard R-matrix, satisfying the Yang-Baxter relation, actually intertwines the coproduct of a Hopf algebra with its opposite.

Proposition 12.1. (a) There exists a unique homomorphism $\Delta:\ \mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$, such that

$$\begin{aligned}\Delta(H_a)&=H_a\otimes 1+1\otimes H_a\ ,\quad a\in M\ ,\\ \Delta(e_\alpha)&=1\otimes e_\alpha+e_\alpha\otimes e^{\varphi(\alpha,\cdot)}\ ,\\ \Delta(e_{-\alpha})&=e^{-\varphi(\cdot,\alpha)}\otimes e_{-\alpha}+e_{-\alpha}\otimes 1\ ,\quad \alpha\in N\ .\end{aligned}\tag{12.1}$$

(b) If $I\subset\mathcal{A}$ is the ideal generated by the constants in $\mathcal{A}^+$ and $\mathcal{A}^-$, and $\mathcal{A}'=\mathcal{A}/I$, then Δ induces a unique homomorphism $\mathcal{A}'\to\mathcal{A}'\otimes\mathcal{A}'$ that will also be denoted Δ, so that (12.1) holds with H_a and $e_{\pm\alpha}$ being interpreted as generators of $\mathcal{A}/I$.

(c) Let Δ' be the opposite coproduct on $\mathcal{A}/I$, and R the standard R-matrix on $\mathcal{A}/I$ (satisfying Yang-Baxter), then $\Delta R=R\Delta'$.

(d) The algebra $\mathcal{A}$ becomes a Hopf algebra when endowed with the counit $\mathcal{E}$ and the antipode S. The former is the unique homomorphism $\mathcal{A}\to\mathbb{C}$ such that

$$\begin{aligned}&\mathcal{E}(a)=1\ ,\quad \mathcal{E}(H_a)=0\ ,\quad a\in M\ ,\\ &\mathcal{E}(e_{\pm\alpha})=0\ ,\quad \alpha\in N\ .\end{aligned}\tag{12.2}$$

The antipode is the unique anti-automorphism $S:\ \mathcal{A}\to\mathcal{A}$ such that

$$\begin{aligned}&S(1)=1\ ,\quad S(H_a)=-H_a\ ,\quad a\in M\ ,\\ &S(e_\alpha)=-e_\alpha e^{-\varphi(\alpha,\cdot)}\ ,\quad S(e_{-\alpha})=-e^{\varphi(\cdot,\alpha)}e_{-\alpha}\ ,\quad \alpha\in N\ .\end{aligned}\tag{12.3}$$

(e) The counit $\mathcal{E}$ and the antipode S of $\mathcal{A}$ induce analogous structures on $\mathcal{A}'=\mathcal{A}/I$ such that (12.3) holds on $\mathcal{A}'$.

Proof.

(a) The verification amounts to checking that $\Delta(\mathcal{A})$ has the relations of $\mathcal{A}$, in particular,

$$\big[\Delta(e_\alpha),\Delta(e_{-\beta})\big]=\delta_\alpha^\beta\ \Delta\big([e_\alpha,e_{-\beta}]\big)\ .$$

(b) The ideal I is generated by elements $x\in\mathcal{A}^+$ and $y\in\mathcal{A}^-$ such that $[e_{-\alpha},x]=0=[e_\alpha,y],\ \alpha\in N$. Since $\Delta:\ \mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ is a homomorphism, Δ induces a homomorphism

$\mathcal{A}/I \to (\mathcal{A}\otimes\mathcal{A})/\Delta(I)$. We must show that $\Delta(I) \subset I\otimes\mathcal{A}+\mathcal{A}\otimes I$. Since I is generated by elementary constants, it is enough to show that, for an elementary constant C, $\Delta(C) \subset I\otimes\mathcal{A}+\mathcal{A}\otimes I$. Let $C \in \mathcal{A}^+$ be an elementary constant; then $[e_{-\alpha}, C] = 0$ and thus $[\Delta(e_{-\alpha}), \Delta(C)] = 0,\ \alpha \in N$. If C is of order n in the generators, then (12.1) shows that

$$\Delta C = 1\otimes C + P^1 \otimes P_{n-1} + P^2 \otimes P_{n-2} + \ldots + C \otimes P_0 \ ,$$

where P^n and P_k are homogeneous of order k in the e_α's. Because C is an elementary constant—Definition 4.1.— there is no constant among the $P^k, P_k,\ n = 1, \ldots, n-1$; then $[\Delta(e_{-\alpha}), \Delta(C)] = 0$ implies that $\Delta C = 1\otimes C + C\otimes P_0$ which indeed belongs to $I\otimes\mathcal{A}+\mathcal{A}\otimes I$; consequently Δ provides a map $\mathcal{A}/I \to \mathcal{A}/I \otimes \mathcal{A}/I$.

(c) We use the abbreviation – compare (10.6), Definition 2.2 and Eq. (2.9),

$$R = t^{(\alpha')}_{(\alpha)} R^i[e_{-\alpha}] \otimes R_i[e_{\alpha'}] \ ,$$

$$\begin{aligned}\Delta(e_\beta)R - R\Delta'(e_\beta) = t^{(\alpha')}_{(\alpha)}\big(&R^i[e_{-\alpha}] \otimes e_\beta R_i[e_{\alpha'}] \\ &+ e_\beta R^i[e_{-\alpha}] \otimes e^{\varphi(\beta,\cdot)} R_i[e_{\alpha'}] - R^i[e_{-\alpha}]e_\beta \otimes R_i[e_{\alpha'}] \\ &- R^i[e_{-\alpha}]e^{\varphi(\beta,\cdot)} \otimes R_i[e_{\alpha'}]e_\beta\big) \ .\end{aligned}$$

Terms 2 and 3 combine to $R^i[e_\beta, t^{(\alpha')}] \otimes R_i[e_{\alpha'}]$, and the recursion relations (7.2) implies that the sum of all four terms is equal to zero.

(d) The existence and the uniqueness of the homomorphism $\mathcal{E}$ and the anti-homomorphism S are obvious. We have to show that $\mathcal{E}$ satisfies the axioms

$$(\mathcal{E} \times id)\Delta = id = (id \times \mathcal{E})\Delta \ ,$$

which is straightforward, and that

$$m(id \times S)\Delta = \epsilon = m(S \times id) \ .$$

Here m indicates multiplication, $\mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$. For example,

$$m(id \times S)\Delta(e_\alpha) = S(e_\alpha) + e_\alpha e^{-\varphi(\alpha,\cdot)} = 0 \ .$$

(e) Obvious, since $\mathcal{E}(I) = 0$ and $S(I) = I$ by Proposition 7.2. Theorem 12 is proved.

We turn to the case of the deformed R-matrix of Section 10, all statements should be understood to hold to first order in the deformation parameter ϵ. The maps $\Delta, \mathcal{E}$ and S are as before and the deformed maps are $\Delta_\epsilon = \Delta + \epsilon\Delta_1$, $\quad \mathcal{E}_\epsilon = \mathcal{E} + \epsilon\mathcal{E}_1$, $\quad S_\epsilon = S + \epsilon S_1$.

Proposition 12.2. (a) There is a unique homomorphism $\Delta_\epsilon : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that

$$\Delta_1(x) = [\Delta(x),\ Ke_{-\rho} \otimes Ke_\sigma] , \quad x \in \mathcal{A} .$$

(b) The projection of Δ_ϵ to $\mathcal{A}' \to \mathcal{A}' \otimes \mathcal{A}'$ is well defined. (c) Let Δ'_ϵ be the opposite coproduct on $\mathcal{A}' = \mathcal{A}/I$, and $R_\epsilon = R + \epsilon R_1$ the R-matrix of Theorem 10, then $\Delta_\epsilon R_\epsilon = R_\epsilon \Delta'_\epsilon$ (to first order in ϵ). (d) The deformed counit and antipode of $\mathcal{A}$ are given by $\mathcal{E}_1 = 0$ and

$$S_1(x) = [Ke_{-\rho}e_\sigma,\ S(x)] , \quad x \in \mathcal{A} .$$

(e) The counit $\mathcal{E}_\epsilon$ and the antipode S_ϵ induce analogous structures on $\mathcal{A}/I$.

Proof. (a) By the Jacobi identity. (b) Obvious, for $\Delta_1(C) = [\Delta(C),\ Ke_{-\rho} \otimes Ke_\sigma] \in I \otimes \mathcal{A} + \mathcal{A} \otimes I$. (c) Completely straightforward. (d) We have $(\mathcal{E} \times id)\Delta_1(x) = 0$, whence $\mathcal{E}_1 = 0$, while

$$m(id \times S_1)\Delta(H_a) + m(id \times S)\Delta_1(H_a) = 0$$

since

$$\begin{aligned} m(id \times S_1)\Delta(H_a) &= S_1(H_a) = [H_a,\ Ke_{-\rho}e_\sigma] , \\ m(id \times S)\Delta_1(H_a) &= m(id \times S)(H_a(\sigma) - H_a(\rho)Ke_{-\rho} \otimes Ke_\sigma \\ &= \big(H_a(\sigma) - H_a(\rho)\big)Ke_{-\rho}\big(-e_\sigma e^{-\varphi(\sigma,\cdot)}K^{-1}\big) \\ &= -\big(H_a(\sigma) - H_a(\rho)\big)Ke_{-\rho}e_\sigma = -[H_a, Ke_{-\rho}e_\sigma] \end{aligned}$$

and

$$m(id \times S_1)\Delta(e_\alpha) + m(id \times S)\Delta_1(e_\alpha) = 0$$

since

$$\begin{aligned} m(id \times S_1)\Delta(e_\alpha) &= S_1(e_\alpha) + e_\alpha S_1(e^{\varphi(\alpha,\cdot)}) \\ &= S_1(e_\alpha) - e_\alpha\big[e^{-\varphi(\alpha,\cdot)},\ Ke_{-\rho}e_\sigma\big] \\ &= [e_\alpha, Ke_{-\rho}e_\sigma]e^{-\varphi(\alpha,\cdot)} . \\ m(id \times S)\Delta_1(e_\alpha) &= -[e_\alpha, Ke_{-\rho}e_\sigma]e^{-\varphi(\alpha,\cdot)}. \end{aligned}$$

These last two results require some work.

(e) This is clear, since $\mathcal{E}_1 = 0$ and $S_1(I) \in I$. The theorem is proved.

References.

[BD] A.A. Belavin and V.G. Drinfeld, Sov.Sci.Rev.Math.**4** (1984) 93-165.

[C] C. Chevalley, "Sur la classification des algebres de Lie simples et de leur representations. C.R. **227** (1948) 1136-1138.

[D1] V.G. Drinfeld, in Proceedings of the International Congress of Mathematicians, Berkeley A.M. Gleason, ed. (American Mathematical Society, Providence RI 1987.)

[D2] V.G. Drinfeld, in "Quantum Groups," P.P. Kulish, ed. Proceedings of the Workshop held in the Euler International Mathematical Institute, Leningrad, Fall 1990.

[CG] E. Cremmer and J.-L. Gervais, Commun.Math.Phys. **134** (1990) 619.

[F] C. Frønsdal, in Proceedings of the Karpac Winter School, February 1994.

[FG1] C. Frønsdal and A. Galindo, Contemporary Mathematics, **175** (1994) 73-88.

[FG2] C. Frønsdal and A. Galindo, Lett.Math.Phys. **34** (1995) 25-36.

[K] V.G. Kac, "Infinite Dimensional Lie Algebras," Cambridge University Press 1990.

[KR] A.N. Kirillov and N.Yu. Reshetikhin, Commun.Math.Phys. **134** (1991) 421-431.

[Ma] Yu.I. Manin, "Topics in noncommutative geometry," Princeton University Press, Princeton NJ 1991.

[M] R.V. Moody, "A new class of Lie algebras." J. Algebra **10** (1968) 211-230.

[LSo] J. Geom. Phys. **7** (1991) 1-14.

[R] N.Yu. Reshetikhin, Lett.Math.Phys. **20** (1990) 331-336.

[Ro] M. Rosso, Commun.Math.Phys.**124** (1989) 307-319.

[Sc] A. Schirrmacher, Z. Phys.C **50** (1991) 321.

[Su] A. Sudbery, J.Phys.A.Math.Gen. **23** (1990) L697.

[V] V.S. Varadarajan